The Maxwell Source Equations

using quaternions operators

Eq#

$$\frac{1}{4}\left(\nabla A - (\nabla A)^*\right)\left(A\nabla - (A\nabla)^*\right) = \left(0, \nabla_0 A + \nabla_u \phi + \nabla \times A\right)\left(0, \nabla_0 A + \nabla_u \phi - \nabla \times A\right) \qquad 1.1$$

$$= \left(0, -E + B\right)\left(0, -E - B\right) = \left(B^2 - E^2, 2E \times B\right) \qquad 1.2$$

$$\mathcal{L}_{EB} \equiv \frac{1}{4}\left(\left(B^2 - E^2, 2E \times B\right) + \left(B^2 - E^2, 2E \times B\right)^*\right) - \frac{1}{2}\left(\left(JA + (JA)^*\right)\right) \qquad 1.3$$

$$= \frac{1}{2}\Big(-(\nabla_1 \phi)^2 - (\nabla_2 \phi)^2 - (\nabla_3 \phi)^2 - (\nabla_0 A_1)^2 - (\nabla_0 A_2)^2 - (\nabla_0 A_3)^2$$

$$+ (\nabla_3 A_2)^2 + (\nabla_2 A_3)^2 + (\nabla_1 A_3)^2 + (\nabla_3 A_1)^2 + (\nabla_2 A_1)^2 + (\nabla_1 A_2)^2\Big) - \rho\phi + J_1 A_1 + J_2 A^2 + J_3 A_3$$

$$- (\nabla_3 A_2)(\nabla_2 A_3) - (\nabla_1 A_3)(\nabla_3 A_1) - (\nabla_1 A_2)(\nabla_2 A_1) - (\nabla_1 \phi)(\nabla_0 A_1) - (\nabla_2 \phi)(\nabla_0 A_2) - (\nabla_3 \phi)(\nabla_0 A_3)$$

Calculate the field equations

$$\nabla_\mu\left(\frac{\partial \mathcal{L}_{EB}}{\partial(\nabla_\mu \phi)}\right) = -\nabla_1^2 \phi - \nabla_2^2 \phi - \nabla_3^2 \phi - \nabla_0 \nabla_1 A_1 - \nabla_0 \nabla_2 A_2 - \nabla_0 \nabla_3 A_3 - \rho = \nabla \cdot E - \rho = 0 \qquad 1.4$$

$$\nabla_\mu\left(\frac{\partial \mathcal{L}_{EB}}{\partial(\nabla_\mu A_1)}\right) = -\nabla_0^2 A_1 + \nabla_3^2 A_1 + \nabla_2^2 A_1 - \nabla_1 \nabla_3 A_3 - \nabla_1 \nabla_2 A_2 - \nabla_0 \nabla_1 \phi - J_1 = \nabla_0 E_1 - (\nabla \times B)_1 + J_1 = 0$$

$$\nabla_\mu\left(\frac{\partial \mathcal{L}_{EB}}{\partial(\nabla_\mu A_2)}\right) = -\nabla_0^2 A_2 + \nabla_3^2 A_2 + \nabla_1^2 A_2 - \nabla_2 \nabla_3 A_3 - \nabla_1 \nabla_2 A_1 - \nabla_0 \nabla_2 \phi - J_2 = \nabla_0 E_2 - (\nabla \times B)_2 + J_2 = 0$$

$$\nabla_\mu\left(\frac{\partial \mathcal{L}_{EB}}{\partial(\nabla_\mu A_3)}\right) = -\nabla_0^2 A_3 + \nabla_2^2 A_3 + \nabla_1^2 A_3 - \nabla_2 \nabla_3 A_2 - \nabla_1 \nabla_3 A_1 - \nabla_0 \nabla_3 \phi - J_3 = \nabla_0 E_3 - (\nabla \times B)_3 + J_3 = 0$$

$$\boxed{\begin{aligned} \nabla \cdot E &= \rho \qquad 1.8 \\ \nabla \times B - \nabla_0 E &= J \qquad 1.9 \end{aligned}}$$

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